Product

[illegible]

$$\sum u_k = 1 \quad , \quad \sum_k v_k = 1$$

[illegible] $u_k * v_k = 1$!

[illegible]

$$\sum_{m=0} \sum_{k=0} u_k v_{n-k}$$

[illegible]

This space intentionally left blank.

	a_0	a_1	a_2	$\cdots$	a_n
b_0	1	x	x^2		x^n
b_1	x	x^2			
b_2	x^2				
$\vdots$					
b_n	x^n				

Proof that Cauchy product converges to the product.

The proof proceeds in 3 parts:

1) We show that for a convergent series, the average of its partial sums converges to the sum of the series.
2) We show [illegible] product is convergent.
3) We show that the average of the partial sums of the Cauchy product is equal to the product [illegible] of [illegible] partial sums [illegible].

1) Average of partial sums converges to [illegible] sum.

$$S = \sum_{k=0}^{\infty} x_k \ , \quad S_n = \sum_{k=0}^{n} x_k \ , \quad \sigma_n = \frac{1}{n+1}\sum_{k=0}^{n} S_k \Rightarrow \sigma_n \to S$$

series x_n converges to S — partial sums — average of partial sums — average of partial sums [illegible]

Note that σ_n is an infinite sequence, we want to show that this sequence converges to the sum S.

Start by subtracting S from σ_n:

$$\sigma_n - S = \left(\frac{1}{n+1}\sum_{k=0}^{n} s_k\right) - S\left(\frac{n+1}{n+1}\right) = \frac{1}{n+1}\sum_{k=0}^{n}(s_k - S)$$

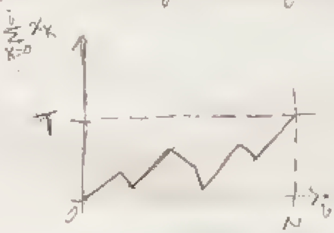
$\longrightarrow$

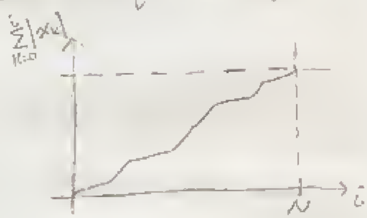
$$\sigma_n - S = \frac{1}{n+1} \sum_{k=0}^{n} (S_k - S)$$

Now lets look at what happens when we take the absolute value [illegible] [illegible] in the sum.

Imagine a sum in general. $T = \sum_{k=0}^{N} t_k$

If any of the values t_k are negative then the sum will move up & down [illegible] the terms [illegible] [illegible]. If instead we add up the absolute values of the terms, $\sum_{k=0}^{N} |t_k|$, the sum will increase [illegible] as the terms are added up. That means the sum can't be less than T, because we're no longer adding any negatives along the way.





In [illegible]

$$\sum_{k=0}^{N} x_k \leq \sum_{k=0}^{N} |x_k|$$

Furthermore, the magnitude of T cannot exceed $\sum_{k=0}^{N} |x_k|$. To see this, [illegible] [illegible] the sum [illegible] [illegible] of the sum [illegible] [illegible] of [illegible] [illegible] by adding up all negative values, or by adding up all positive values. Either way, [illegible] of the sum [illegible] [illegible] But the latter case [illegible] [illegible] already [illegible] $\sum_{k=0}^{N} |x_k|$, so in both cases, the magnitude of the sum is equal to $\sum_{k=0}^{N} |x_k|$. Therefore

$$|T| \leq \sum_{k=0}^{N} |x_k|$$

Now that we know that:

$$\sigma_n - S = \frac{1}{n+1}\sum_{k=0}^{n}(S_k - S) \implies |\sigma_n - S| \leq \frac{1}{n+1}\sum_{k=0}^{n}|S_k - S|$$

Now, because S_k is the k-th partial sum of a series that sums to S, then the infinite sequence S_k converges to S. More precisely, for any value $\varepsilon > 0$, we can find some index M such that for all $k > M$, the difference between S_k & S is less than ε:

$$|S_k - S| < \varepsilon \text{ , for } k > M.$$

So we can now split our sum at M:

$$|\sigma_n - S| \leq \frac{1}{n+1}\left(\sum_{k=0}^{M}|S_k - S| + \sum_{k=M+1}^{n}|S_k - S|\right)$$

Looking at the second sum, since all of the terms are at indices greater than M, so the terms are all less than ε:

$$|\sigma_n - S| < \frac{1}{n+1}\left(\sum_{k=0}^{M}|S_k - S| + \sum_{k=M+1}^{n}\varepsilon\right) = \frac{1}{n+1}\sum_{k=0}^{M}|S_k - S| + \frac{n-M}{n+1}\varepsilon$$

Now lets look at the first sum, from 0 to M. This sum is a fixed constant: no matter how large we let n get (as we go in the sequence $|\sigma_n - S|$), M is fixed (by our choice of ε) so the finite sum is constant, which we call L:

$$L = \sum_{k=0}^{M}|S_k - S|$$

$$\therefore \quad |\sigma_n - S| < \frac{L}{n+1} + \frac{n-M}{n+1}\varepsilon < \frac{L}{n+1} + \varepsilon$$

[illegible] part [illegible] because $\frac{n-M}{n+1}$ always [illegible] 1.

→

[illegible],

$$|a_n - S| < \frac{L}{n+1} + \varepsilon$$

Now we are trying to show that the [illegible] sequence a_n converges to S. [illegible] [illegible] [illegible] [illegible]

We [illegible] [illegible] $\varepsilon > 0$, & [illegible] an appropriate index M, & [illegible] [illegible] so for any value $\varepsilon_2 > 0$, we can pick a value $\varepsilon < \varepsilon_2$ and then all we need to do is go [illegible] [illegible] the sequence, say to some index N, such that $\frac{L}{N+1}$ is less than $\varepsilon_2 - \varepsilon$:

$$\frac{L}{N+1} + \varepsilon < \varepsilon_2$$

And [illegible]:

$$|a_N - S| < \varepsilon_2$$

And so, we have shown that for any chosen limit, $\varepsilon_2 > 0$, we [illegible] choose an N such that the difference between a_N & S is less than ε_2, & therefore a_n converges to S. QED.

This completes part 1 of the Cauchy product proof.

Part 2 – Proof that Cauchy Product converges.

[illegible] [illegible] of [illegible] Cauchy product,

$$\sum_{n=0}^{\infty} \sum_{k=0}^{n} u_k v_{n-k}$$

So if [illegible] define a sequence $c_k = \sum_{j=0}^{k} u_k v_{j-k}$, [illegible] the Cauchy product is $\sum_{n}^{\infty} c_n$.

So we want to show that $\sum_{n=0}^{\infty} c_n$ converges which means the Cauchy product converges.

We'll look at absolute convergence, $\sum_{n=0}^{\infty} |c_n|$. Recall that (as we showed a few pages ago, pg 79):

$$\left|\sum_{n=0}^{N} x_n\right| \leq \sum_{n=0}^{N} |x_n|$$

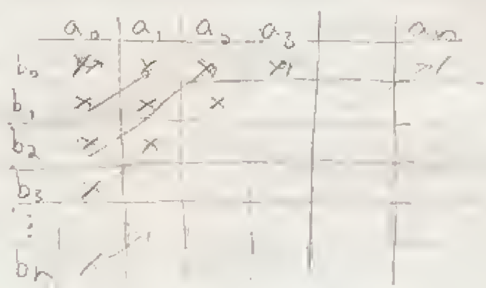
So since $c_n = \sum_{k=0}^{n} u_k v_{n-k}$, $|c_n| \leq \sum_{k=0}^{n} |u_k v_{n-k}|$, and so:

$$\sum_{n=0}^{\infty} |c_n| \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}|$$

And likewise $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} |c_n|$ so $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}|$

For convenience, we will define $a_k = |u_k|$, & $b_k = |v_k|$.
Therefore $\sum_{n=0}^{\infty} c_n \leq \sum_{n=0}^{\infty} \sum_{k=0}^{n} |u_k v_{n-k}| = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$.

The sum of these products traces a triangle in the a×b grid:



All the marked products are summed together, and the resulting sum is no less than the Cauchy product.

[illegible], if [illegible] the [illegible] grows, it will not be less from the [illegible] so, [illegible] product:

$$\sum_{n=0}^{\infty}|c_n| \leq \sum_{n=0}^{\infty}\sum_{k=0}^{n} a_k b_{n-k} \leq \sum_{n=0}^{\infty}\left(a_n \sum_{k=0}^{\infty} b_k\right)$$

sum of all [illegible]

[illegible] $(a_0 b_0 + a_0 b_1 + a_0 b_2 + \ldots)$
$+ (a_1 b_0 + a_1 b_1 + a_1 b_2 + \ldots)$

[illegible] of each [illegible]

$a_0 [illegible]$
$+ a_1 [illegible]$
$\vdots$

And [illegible] of it.

[illegible]

$\sum_{k=0}^{\infty} b_k$, $\sum_{k=0}^{\infty} a_k$ [illegible].

$$\therefore \sum_{n=0}^{\infty}|c_n| \leq \sum_{k=0}^{\infty} a_k \sum_{k=0}^{\infty} b_k \leq A\,B$$

Where $A = \sum_{k=0}^{\infty} a_k$ & $B = \sum_{k=0}^{\infty} b_k$. Since $a_k = |u_k|$ and $b_k = |v_k|$, and since u_k & v_k are absolutely convergent, A & B are finite, and so $\sum_{n=0}^{\infty}|c_n|$ is bounded above, and therefore $\sum_{n=0}^{\infty} c_k$ (the Cauchy Product) is absolutely convergent. QED.

This concludes the [illegible] proof.

Part 3 — Verify that the average of the partial sums of the Cauchy product converges to the product.

Recall that:

$$U = \sum_{k=0}^{\infty} u_k \quad , \quad V = \sum_{k=0}^{\infty} v_k$$

And we'll call the Cauchy product W:

$$W = \sum_{k=0}^{\infty} w_k \quad ; \quad w_k = \sum_{j=0}^{k} u_j v_{k-j}$$

And we have the partial sums:

$$U_n = \sum_{k=0}^{n} u_k \quad ; \quad V_n = \sum_{k=0}^{n} v_k \quad ; \quad W_n = \sum_{k=0}^{n} w_k = \sum_{k=0}^{n} \sum_{j=0}^{k} u_j v_{k-j}$$

We know, as a property of infinite series, that the sequence of partial sums converges to the sum of the series:

$$U_n \to U \quad ; \quad V_n \to V \quad ; \quad W_n \to W$$

We now want to show that the Cauchy product, W, is equal to the product of the two series' sums:

$$W = UV$$

This is, after all, what we're trying to prove overall.

We start w/ the average the first $m+1$ partial sums of the Cauchy product:

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} W_n$$

$\longrightarrow$

[illegible] as [illegible] [illegible] ... we're going to show that

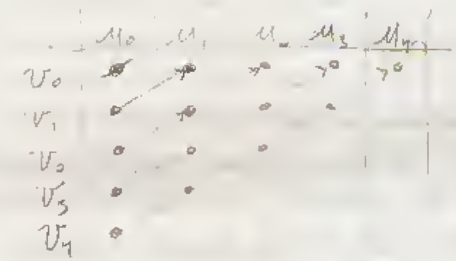
$$S_m := \sum_{n=0}^{m} W_n = \sum_{n=0}^{m} U_n V_{m-n}$$

So that we can [illegible] average σ_m in terms of U_k & V_k.

First let's look at W_n. This is a partial sum of the [illegible] product $W_n = (u_0 v_0) + (u_0 v_1 + u_1 v_0) + \ldots + (u_0 v_m + \ldots + u_m v_0)$

[illegible] each of these partial sums, W_n, is the sum of $n+1$ diagonal rows in the product [illegible] of u_k, v_k:

$W_4 =$

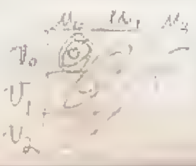


Marked products are summed to W_n. Each diagonal [illegible] [illegible] progression [illegible] [illegible] summation of the [illegible] summation of W_n.

So the first few partial sums [illegible] like

[illegible], [illegible], W_0 [illegible], W_1 [illegible]

[illegible] [illegible] [illegible] we're adding up a bunch of these partial products together in S_m, we're actually adding [illegible] the [illegible] $(u_0 v_0)$ multiple & the second row $(u_0 v_1 + u_1 v_0)$ [illegible] times, etc:



Now we'll look at the right side: $\sum_{n=0}^{m} U_n V_{m-n}$

U_n is a partial sum of u_k: $U_0 = u_0$, $U_1 = u_0 + u_1$, $U_2 = u_0 + u_1 + u_2$, etc., & likewise with V_n, except that the partial sums of v_k are going in the other direction.

So this sum of products of partial sums looks like this:

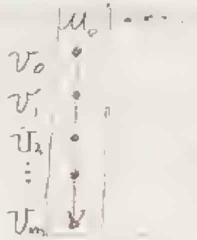
$$\sum_{n=0}^{m} U_n V_{m-n} =$$

$(n=0,\ U_0 V_m)$ $\quad u_0 (v_0 + v_1 + \cdots + v_m)$
$(n=1,\ U_1 V_{m-1})$ $\quad + (u_0 + u_1)(v_0 + \cdots + v_{m-1})$
$(n=2)$ $\quad + (u_0 + u_1 + u_2)(v_0 + \cdots + v_{m-2})$
$\vdots$
$(n=m-1)$ $\quad + (u_0 + \cdots + u_{m-1})(v_0 + v_1)$
$(n=m)$ $\quad + (u_0 + \cdots + u_m)(v_0)$

Each term, $U_n V_{m-n}$, in this sum also adds up some products in the u_k, v_k product grid. In this case, each term doesn't cover a triangle with sides of length m, but instead a rectangle of dimensions $(n+1)\times(m+1-n)$:

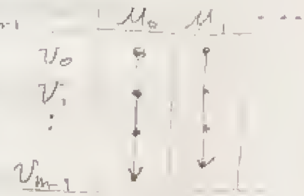
$n=0$: $U_0 V_m$

$1\times(m+1)$



$n=1$: $U_1 V_{m-1}$

$2\times m$



$n=2$: $U_2 V_{m-2}$

$3\times(m-1)$

$n=3$: $U_3 V_{m-3}$

$4\times(m-2)$

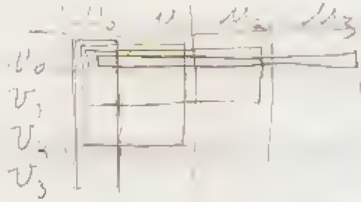
And finally:

$n=m: u_m v_0$

$(m+1) \times 1$

So now we add all the rectangles together:

$\sum_{n=0}^{3} u_n v_{3-n}$:



And so we've got the same thing: the top left corner, $u_0 v_0$, is included in all $m+1$ rectangles. Each product in the second diagonal, $(u_0 v_1 + u_1 v_0)$ are each included in all but 1 rectangle ($u_0 v_1$ is in all but the last; $u_1 v_0$ is in all but the first), & so they're each summed m times. Each product in the third diagonal is included in all but 2 rectangles, & so are each included in the sum $m-1$ times, etc.

And so we see that

$$\sum_{n=0}^{m} W_n = \sum_{n=0}^{m} u_n v_{m-n}.$$

And we can therefore say that the average of our partial sums is

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} u_n v_{m-n}$$

Now we define α_k & β_k to be the error terms between the kth partial sum & the series sum for u_k & v_k, respectively.

$$U_n = U + \alpha_n \quad , \quad V_n = V + \beta_n$$

And so

$$\sigma_m = \frac{1}{m+1} \sum_{n=0}^{m} (U + \alpha_n)(V + \beta_{m-n})$$

$$= \frac{1}{m+1} \sum_{n=0}^{m} \left(UV + U\beta_{m-n} + V\alpha_n + \alpha_n \beta_{m-n} \right)$$

$$= \frac{1}{m+1} UV \sum_{n=0}^{m} 1 + \frac{1}{m+1} U \sum_{n=0}^{m} \beta_{m-n} + \frac{1}{m+1} V \sum_{n=0}^{m} \alpha_n + \frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$$

$$= UV + \frac{U}{m+1} \sum_{n=0}^{m} \beta_n + \frac{V}{m+1} \sum_{n=0}^{m} \alpha_n + \frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$$

The first term comes because we're adding up $(m+1)$ 1's, & multiplying by $\frac{1}{m+1}$. Notice we changed β_{m-n} to β_n in the second term. That's ok, all we did was add up the series backwards.

So we're looking at an infinite sequence in σ_m. We know that the error terms, α_n & β_n, both go to zero as n goes to infinity, & so these two series: $\Sigma\alpha_n$ & $\Sigma\beta_n$, both converge, i.e., the sum is finite. Which means that the entire second & third terms can be driven arbitrarily close to zero by choosing a high enough m (going far enough out into the σ_m sequence).

Now we look at the final term $\frac{1}{m+1} \sum_{n=0}^{m} \alpha_n \beta_{m-n}$.

We know the the error terms are each bounded above, because the series of which they are the error terms converge. We'll call M the maximum of α_k & β_k for all k.

→

In other words:

$$|\alpha_k| < M, \quad |\beta_k| < M \quad \text{for all } k.$$

Also, since α_k & β_k both converge to zero, we can also say that for any value $\varepsilon > 0$, there is an N such that:

$$|\alpha_k| < \varepsilon, \quad |\beta_k| < \varepsilon, \quad \text{for } k > N.$$

Now we're going to show that this final term converges absolutely to zero. Note that this forms a sequence indexed by m:

$$\left|\frac{1}{m+1}\sum_{n=0}^{m} \alpha_n \beta_{m-n}\right| \leq \frac{1}{m+1}\sum_{n=0}^{m} \left|\alpha_n \beta_{m-n}\right|$$

(we proved this back in part I).

Now we can split up our sum:

$$= \frac{1}{m+1}\left(\sum_{n=0}^{N} \alpha_n \beta_{m-n} + \sum_{n=N+1}^{m} \alpha_n \beta_{m-n}\right)$$

$$\leq \frac{1}{m+1}\left((N+1)(M^2) + (m-N)\,\varepsilon^2\right)$$

$$= \frac{N+1}{m+1} M^2 + \frac{m-N}{m+1}\varepsilon^2 = \frac{N+1}{m+1}M^2 + \frac{m}{m+1}\varepsilon^2 - \frac{N}{m+1}\varepsilon^2$$

$$< \frac{N+1}{m+1} M^2 - \frac{N}{m+1}\varepsilon^2$$

Now N, M, & ε are all constants so this goes to zero as m goes to infinity, therefore that final term, $\frac{1}{m+1}\sum_{n=0}^{m}\alpha_n\beta_{m-n}$, converges absolutely to zero.

And so going back to the last page, we have

$$\sigma_m \to UV \text{ as } m \text{ goes to infinity. QED.}$$

This concludes the third & final part of the proof.

So in Summary: The Cauchy product is absolutely convergent, & the average of its partial sums converges to the product UV. We showed in part 1 that the average of partial sums of an absolutely convergent series is equal to the sum of the series.

Therefore, the Cauchy Product is equal to UV. QED

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